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Filtering of non-linear instabilities*

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SUMMARY

For Courant numbers larger than one and cell Reynolds numbers larger than two, oscillations and in some cases instabilities are typically found with implicit numerical solutions of the fluid dynamics equations. This behavior has sometimes been associated with the loss of diagonal dominance of the coefficient matrix. It is shown here that these problems can in fact be related to the choice of the spatial differences, with the resulting instability related to aliasing or non-linear interaction. Appropriate "filtering" can reduce the intensity of these oscillations and in some cases possibly eliminate the instability. These filtering procedures are equivalent to a weighted average of conservation and non-conservation differencing. The entire spectrum of filtered equations retains a three-point character as well as second-order spatial accuracy. Burgers equation has been considered as a model. Several filters are examined in detail, and smooth solutions have been obtained for extremely large cell Reynolds numbers.

1. Introduction

Three-point finite-difference discretization has generally been used to approximate the partial differential equations of fluid mechanics. Explicit formulations are typically restricted by the 'linear' stability conditions

$$c = \bar{u}\Delta t/\Delta x \leq 1 \text{ and } R_c = \bar{u}\Delta x/\nu \leq 2, \tag{1}$$

where c is the Courant number; R_c is the cell Reynolds number; $\bar{u}$ is a reference velocity; Δt, Δx are the temporal and spatial increments, respectively.

Implicit central difference formulations designated by (ICD) are linearly unconditionally stable; however, the diagonal dominance of the tridiagonal inversion matrix is assured only if conditions (1) are satisfied. If conditions (1) are violated, i.e., there is a loss of diagonal dominance, error growth is possible and spurious oscillations are observed [1]. Recently a difference approximation, denoted by (KR), that insures linear stability and still maintains diagonal dominance, even when conditions (1) have been violated has been proposed [2]. For linear systems, solutions have been obtained for $c \gg 1$ and $R_c \gg 2$. Unfortunately, it has been shown that the KR diagonally dominant formulation also exhibits instabilities for the

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non-linear Burgers equation, when conditions (1) are violated; moreover, for the same conditions, stable solutions are sometimes obtained with the non-diagonally dominant central differences [3]. Therefore, it may be concluded that *non-linear instability (aliasing) and not diagonal dominance is the primary reason for error growth when conditions (1) are violated*. This may explain the difficulties for $c >> 1$ that have been encountered with implicit Navier-Stokes finite-difference solvers [3]. Since lack of diagonal dominance leads to spurious oscillations [1], for non-linear systems this enhances the aliasing effect.

Previous studies on non-linear instability by Phillips [4], Arakawa [5], Piacsek and Williams [6] and others have been presented for explicit schemes; these satisfy the CFL condition $c < 1$. However, the lack of conservation of certain quadratic quantities over the whole domain still leads to an aliasing error growth. A smoothing procedure, or alternatively, proper spatial differencing (filtering) can help to eliminate this instability. In the past, first-order accurate upwind differencing has been used extensively, as it has sufficient numerical viscosity to suppress these instabilities. This corresponds to a severe filter. Other schemes, most often used for hyperbolic systems, e.g., the MacCormack scheme and variants of the Lax-Wendroff method, do not appear to encounter this problem. This is, perhaps, due to the presence of increased amounts of numerical viscosity, either inherent or added artificially to the finite difference system.

In this paper, the Burgers equation is re-evaluated for both the ICD and KR techniques. It is shown that (a) for the linear Burgers equation, with the procedure of Reference [2], there is no instability even if conditions (1) are grossly violated. This confirms the linear analysis of Reference [2] and the earlier results of Reference [3]. Error growth, with the ICD technique, due to the loss of diagonal dominance, is possible but it is shown that this can be avoided in most cases for the linear Burgers equation; (b) for the non-linear Burgers equation, it is shown that instabilities can appear with the ICD and KR formulations, even if the conditions (1) are not simultaneously violated. This would again indicate that diagonal dominance is not responsible for the resulting error growth; (c) for the non-linear Burgers equation, if smoothing or filtering of the non-linear term is applied with the method of Reference [2], there is apparently no instability even if conditions (1) are violated; in addition, spurious oscillations are considerably diminished. The magnitude of the oscillations associated with the lack of diagonal dominance is closely related to the resulting non-linear instability. When these high frequency oscillations are filtered, *even maintaining second-order accuracy*, the aliasing effect is reduced. With a filtered ICD formulation, aliasing can also be controlled; however, the shock may convect to the boundary of the domain. This is true regardless of the boundary location. It will be shown that this solution with the shock located at the boundary is in fact an alternate solution of the ICD difference equations. This non-uniqueness is due to the lack of conservation of $\sum_{j=1}^{N} u_j$ over the domain, and occurs only when the cell Reynolds number $R_c = u\Delta x/(\nu) > 2$.

Finally, the linear as well as non-linear stability of the ICD difference equations is evaluated around a given initial state. This analysis confirms that filtering procedures reduce the magnitude of oscillations and have a positive influence on the non-linear instability. Smoothing procedures of the type utilized by Shuman [7] and investigated in more detail by Shapiro [8] are not considered in the present paper.

2. Analysis

Burgers equation describing a fixed shock wave is given as:

$$u_t + (u - \tfrac{1}{2})u_x = \nu u_{xx}. \tag{2}$$

The initial and boundary conditions are taken to be:

$$\begin{aligned} u(x,0) &= 1 \qquad \text{for } -5 \leq x < 0, \\ u(0,0) &= 0.5, \\ u(x,0) &= 0 \qquad \text{for } 0 < x \leq 5, \end{aligned} \tag{3a}$$

and

$$\begin{aligned} u(x,t) &= 1 \qquad \text{for } x = -5 \\ u(x,t) &= 0 \qquad \text{for } x = 5. \end{aligned} \tag{3b}$$

The exact solution is given to a good approximation in the steady state by:

$$u = \frac{1}{2}\left[1 - \tanh \frac{x}{4\nu}\right].$$

All the solutions discussed in the present paper are obtained with 51 equally spaced grid points; $\Delta x = 0.2$. Convergence was assumed when differences in the values of u between 100 iterations were such that $|u^{k+100} - u^k| < 10^{-6}$; k denotes the iteration number. Many calculations were also run with more severe convergence conditions. The results were unchanged.

Numerous calculations of the linear version of equation (2), where the coefficient of the convective term is treated as a constant, were obtained with both the implicit central differencing (ICD) and the Khosla-Rubin (KR) scheme. The restrictions on cell Reynolds number and Courant number, necessary for diagonal dominance, were grossly violated. In every case a converged solution, for both schemes, was obtained. For the same conditions, with the non-linear Burgers equation, 10 iterations were performed at each time step. For the KR scheme the calculations diverged. The solutions approach a final converged solution to within 10^{-8} and then exhibit a rapid divergence. For cell Reynolds numbers greater than two the solutions obtained with $c < 1$ or for central differences in some cases with $c > 1$, exhibit the expected oscillatory behavior. As the steady state is approached, aliasing effects begin to accumulate, and in most cases for $c > 1$, $R_c > 2$, and even for some where $c < 1$, $R_c > 2$, the solutions diverge for large times. In some of these cases, where R_c is slightly larger than 2, a solution with central differencing can sometimes be obtained, (e.g., $\nu = 1/24$, or $R_c = 2.4$); the ICD scheme has a somewhat larger transient numerical viscosity than does the KR formulation. The above description of linear convergence and non-linear divergence, with equivalent values of c and R_c, indicates that it is probably not the loss of diagonal dominance but the aliasing effect, which is enhanced by spurious oscillations, that is the reason for error growth.

In the following sections, we will show that the source of the aliasing error lies in the form of spatial discretization. For conventional non-conservation form, non-linear instability arises with cell Reynolds numbers greater than two; this result is essentially independent of the value of the Courant number. For explicit schemes, this non-linear instability was first observed by Phillips [4]. One of the remedies he recommended to suppress such error growth was the use of smoothing of the short wave length components. Shuman devised a smoothing operator that is quite often used in numerical weather prediction calculations. Alternatively, modified spatial differencing can suppress the high-frequency modes. The filtering procedure is applied here. Three different filters, including one due to Shuman will be investigated in conjunction with the ICD and KR schemes. The aliasing error growth can then be eliminated so that a steady converged solution is obtained. First-order temporal and second-order spatial accuracy of both the implicit schemes under investigation are retained with all filters.

2.1. *Smoothing and filtering*

In the present section, we will discuss a series of filters that can be used to suppress the aliasing error growth. Only three-point second-order accurate filters are considered. These are given by

$$\tilde{u}_j = \frac{u_{j+1} + u_{j-1} + ku_j}{2+k} . \tag{4}$$

It can be shown that these filters (4) correspond to a weighted average of conservation and non-conservation difference equations. The filters (4) are second-order accurate, with a truncation error $\Delta x^2 u_{xx}/(2+k)$. Several specific cases of interest are:

i) $k = 0$ is conservation form or a trapezoidal filter;
ii) $k = \infty$ is non-conservation form (no filter);
iii) $k = 2$ is the Shuman filter;
iv) $k = -1$ is a special filter to be discussed later.

In general, k can take on any value greater than negative two. However, the optimum choice is governed by the minimum amount of numerical viscosity required to suppress the aliasing error growth. The value $k = -1$ is significant, as will be shown in a later section; for large flow Reynolds number with $k = -1$ the effective cell Reynolds number is always less than or equal to two.

It must be emphasized that there is no general way of arriving at an optimum value of k for any given problem. This can only be obtained by numerical experimentation and some physical insight into the flow characteristics. In addition, the degree of filtering will depend upon the choice of the mesh. For example, for a flat-plate boundary layer governed by the Blasius equation, the non-conservation form of the equations does not lead to stable solutions for grid spacings larger than 2 (see Appendix). Also, for non-uniform grids having large grid spacings in the outer portion of the boundary layer the solution exhibit oscillations and are, in general, rather poor. The use of filtering eliminates many of these problems. With a trapezoidal filter, converged solutions can be obtained with a minimum of grid points within the boundary layer. For non-uniform grids the oscillations in the regions of large mesh size are also reduced or eliminated; however, the second-order numerical viscosity introduced by the filter affects the accuracy of the surface shear stress. This can be reduced by taking a smaller grid near the wall. For moderate uniform grids the artificial viscosity always leads to a less accurate shear stress as compared with the non-conservation solution. Therefore for boundary layers, it would appear that different filters (k values) should be used near and far from the surface.

Once again, it should be stressed that these filters are not applied to smooth the solution after a given time step, but are used to recast the nonlinear coefficient of the u_x term in equation (2). It is in this respect that the present investigation is different from the work of Shuman [7], and similar to that of the Arakawa [5] and William [6] schemes. In what follows, we will examine certain of these filters for two finite-difference schemes, i.e., ICD and KR.

2.1.1 Central differencing (ICD)

Implicit central differencing as is well-known [1] is unconditionally stable. The general form of the difference equations with the filter (4) is as follows:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left[\left(\frac{u_{j+1}^{n+1} + u_{j-1}^{n+1} + ku_j^{n+1}}{2+k}\right) - \frac{1}{2}\right](u_{j+1}^{n+1} - u_{j-1}^{n+1})$$
$$+ \frac{\nu\Delta t}{\Delta x^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}). \qquad (5)$$

In earlier studies, the smoothing character of the conservation difference equations ($k = 0$) has been demonstrated by many investigators [1,3] and will not be repeated here. The resulting oscillations or instability for $R_c > 2$ with non-conservation form ($k = \infty$ or no filter) are also well-documented.

Numerical solutions of (5) have been obtained for $k = \infty$, 0, 2, see Table 1. A typical example is given by the conditions $\Delta t = 6.0$ and $\nu = 1/96$. This corresponds to a Courant number of 15 and a cell Reynolds number of 9.6. The non-conservation equations do not lead to any solution. Although a converged solution is obtained for $k = 0{,}2$, the shock wave is found to convect to the upstream boundary for these conditions. It will be shown that the finite difference equations ((5) with $k = 0$), in fact, has two solutions and that one is consistent with this convective behavior. Since the filter (4) is simply a linear combination of conservation and non-conservation equations, we will investigate the possible non-uniqueness for these forms of the governing equation (5). These are,

$$\frac{du_j}{dt} + \frac{u_{j+1}^2 - u_{j-1}^2}{2} - \frac{u_{j+1} - u_{j-1}}{2} = \frac{1}{R}(u_{j+1} - 2u_j + u_{j-1}), \qquad \text{(conservation)} \qquad (6a)$$

and

$$\frac{du_j}{dt} + (u_j - \tfrac{1}{2})(u_{j+1} - u_{j-1}) = \frac{1}{R}(u_{j+1} - 2u_j + u_{j-1}), \qquad \text{(non-conservation)} \qquad (6b)$$

TABLE I

Solution of Burgers equation R = 9.6

X	K R SOLUTION C > 1 (SLIGHT SHOCK SHIFT)	CONSERVATION FORM C < 1	CONSERVATION FORM C > 1 (SHOCK TO BOUNDARY)	CONSERVATION FORM C > 1 ENFORCED SYMMETRY
-5.0	1.00000	1.00000	1.00000	1.00000
-4.8	1.00000	0.99999	0.20833	1.00000
-4.6	0.99999	1.00000	-0.09336	1.00002
-4.4	1.00000	0.99998	0.07276	0.99989
-4.2	0.99999	1.00002	-0.04184	1.00003
-4.0	1.00001	0.99996	0.02959	0.99997
-3.8	0.99999	1.00006	-0.01838	1.00007
-3.6	1.00002	0.99990	0.01245	0.99992
-3.4	0.99997	1.00014	-0.00798	1.00015
-3.2	1.00005	0.99978	0.00530	0.99978
-3.0	0.99992	1.00032	-0.00344	1.00033
-2.8	1.00012	0.99950	0.00227	0.99951
-2.6	0.99982	1.00076	-0.00148	1.00077
-2.4	1.00027	0.99883	9.722×10^{-4}	0.99884
-2.2	0.99958	1.00177	-6.359×10^{-4}	1.00178
-2.0	1.00064	0.99728	4.171×10^{-4}	0.99729
-1.8	0.99902	1.00411	-2.730×10^{-4}	1.00413
-1.6	1.00150	0.99363	1.789×10^{-4}	0.99365
-1.4	0.99767	1.00954	-1.172×10^{-4}	1.00955
-1.2	1.00353	0.98503	7.682×10^{-5}	0.98504
-1.0	0.99432	1.02194	-5.032×10^{-5}	1.02195
-0.8	1.00844	0.96428	3.297×10^{-5}	0.96429
-0.6	0.98534	1.04977	-2.160×10^{-5}	1.04978
-0.4	1.02107	0.91086	1.415×10^{-5}	0.91087
-0.2	0.95507	1.11056	-9.273×10^{-6}	1.11058
0.0	0.04494	0.50000	6.075×10^{-6}	0.50000

where t is redefined as $t/(2\Delta x)$ and $R = \Delta x/(2\nu)$. The governing equation (2) satisfies the conservation law

$$\frac{d}{dt} \int_{-\infty}^{+\infty} u dx = 0.$$

The discrete forms of Burgers equation (6a) and (6b) should also satisfy this conservation property. Summing over all grid points, we find

$$\frac{d}{dt}\left(\sum_{j=1}^{N} u_j\right) = -\frac{u_2 + u_{N-1} - 1}{2}\left(u_2 - u_{N-1} - \frac{2}{R}\right),$$

(conservation)

and

$$\frac{d}{dt}\left(\sum_{j=1}^{N} u_j\right) = -\left(\frac{1}{2} - \frac{1}{R}\right)(1 - u_2 - u_{N-1}).$$

(non-conservation)

It can be seen that unless the symmetry condition is strictly enforced, i.e. $u_2 + u_{N-1} = 1$, the possibility of a 'spurious' unsymmetric solution, for which $u_2 - u_{N-1} - 2/R = 0$, exists for the conservation form (6a). The solution for which the wave convects to the boundary, in fact, satisfies this latter condition. It has been found, on the other hand, that with a small Δt (Courant number < 1) the symmetric solution is usually recovered. In the next section, it will be shown that for $R \leqslant 2$ the symmetric solution is unique. For $R > 2$, the non-symmetric solution also exists and is also stable. For non-conservation form (6b), only the symmetric solution is possible.

2.1.2 KR scheme

This scheme, as introduced in Reference [2], is diagonally dominant and unconditionally stable for all R and retains second-order spatial and temporal accuracy of the convective derivative. It is given by

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (u_j^{n+1} - \tfrac{1}{2}) \quad (u_j^{n+1} - u_{j-1}^{n+1}) \quad - \frac{\Delta t}{2\Delta x} (\widetilde{u}_j^{n+1} - \tfrac{1}{2}) \, D_j^n + \frac{\nu \Delta t}{\Delta x^2} D_j^{n+1}, \qquad \text{for } u_j > \tfrac{1}{2}, \tag{7a}$$

and

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (u_j^{n+1} - \tfrac{1}{2}) \quad (u_{j+1}^{n+1} - u_j^{n+1}) \quad + \frac{\Delta t}{2\Delta x} (\widetilde{u}_j^{n+1} - \tfrac{1}{2}) \, D_j^n + \frac{\nu \Delta t}{\Delta x^2} D_j^{n+1}, \qquad \text{for } u_j < \tfrac{1}{2}, \tag{7b}$$

where $D_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$.

For the KR formulation, the filters are only used to modify the non-linear coefficient of D_j^n as shown in (7); the implicit portion of the convective term is always given by an upwind differencing form. The resulting equations are second-order accurate in Δx.

Unlike the ICD results, for $\nu = 1/(96)$ and $\Delta t = 6.0$, both the trapezoidal and Shuman filters do lead to converged solutions describing a thin symmetric shock. The results are shown in Table 1. Although the convergence condition is satisfied, there is some indication of a creeping motion of the shock wave. After 3000 time steps this movement was still less than the prescribed tolerance. The infinite time behavior was not considered. For explicit schemes, this problem is virtually undetectable, as the Courant numbers are limited by the CFL stability condition. The allowable Δt is much smaller than that considered here and therefore examination of the large time history would require an extraordinary number of time steps.

2.2 Non-linear and linear stability

In the present section, we examine the stability of the ICD finite difference equivalent of equation (2) about a given initial state. Since the source of aliasing error lies in the spatial discretization, the problem will be examined for a semi-discrete system. The underlying idea is that the spatial differencing leads to a temporal amplification, then there should not be any time discretization that can eliminate this instability. For example, it will be shown, though not rigorously, that for non-conservative form and central differencing, the solution of equation (2) grows exponentially if the cell Reynolds number is greater than two. The stability will be examined for central-differencing for both non-conservative and conservative forms. The effects of other types of filters will also be indicated.

2.2.1 Non-conservation form

With central differencing, equation (2) becomes

$$\frac{du_j}{d\tau} + (u_j - \tfrac{1}{2})\,(u_{j+1} - u_{j-1}) = \frac{1}{R}\,(u_{j+1} - 2u_j + u_{j-1})\,; \tag{8}$$

τ is a non-dimensional time. As it is difficult to find a closed form solution of equation (8) for arbitrary j, only a few simple cases will be examined for small numbers of grid points N.

(i) $N=3$: Since there are only three points, and from the boundary conditions we require $u_1 = 1$ and $u_3 = 0$, then the equation for u_2 becomes

$$\frac{du_2}{d\tau} - (u_2 - \tfrac{1}{2}) = \frac{1}{R}\,(1 - 2u_2)\,. \tag{9}$$

The solution of equation (9) is

$$u_2 = u_2\,(0)\,e^{-\left(\frac{2}{R} - 1\right)t} + \tfrac{1}{2}\left\{1 - e^{-\left(\frac{2}{R} - 1\right)t}\right\}. \tag{10}$$

where $u_2\,(0)$ is the initial value of $u_2(t)$. Clearly the solution diverges for $R > 2$. However, if $u_2(0) = \frac{1}{2}$, as in the present case, Equation (10) leads to $u_2 = \frac{1}{2}$ for all t. However, in this case, there is a cancellation of two growing terms, and therefore the numerical solution can still be amplified by roundoff errors. This is seen to be the case for forward time marching in (9). If we use an explicit method, we obtain

$$u_2^{n+1} = u_2^n\left\{1 + \Delta t - \frac{2\Delta t}{R} + \frac{\Delta t}{2}\left(\frac{2}{R} - 1\right)\right\}\quad . \tag{11}$$

Clearly any small error will grow if the coefficient of u_2^n is greater than unity. Thus, stability is possible only if $R < 2$. For an implicit scheme with the convective coefficient $(u_j - \frac{1}{2})$ treated explicitly, then (9) becomes,

$$u_2^{n+1} = \frac{\frac{2}{R} - 1}{2\left(1 + \frac{2\Delta t}{R}\right)} + \frac{1 + \Delta t}{1 + \frac{2\Delta t}{R}}\, u_2^n \,. \tag{12}$$

Once again, the Neuman stability condition requires $R < 2$ for a stable solution. If, we treat $(u_2 - \frac{1}{2})$ implicitly, the solution converges for almost all Δt and R. However, such a scheme is inconsistent with the differential equation (9).

(ii) $N = 5$: In this case, $u_1 = 1$ and $u_5 = 0$.

$$\begin{aligned} &\frac{du_2}{dt} + (u_2 - \tfrac{1}{2})\,(u_3 - 1) = \frac{1}{R}\,(u_3 - 2u_2 + 1)\,, \\ &\frac{du_3}{dt} + (u_3 - \tfrac{1}{2})\,(u_4 - u_2) = \frac{1}{R}\,(u_4 - 2u_3 + u_2)\,, \\ &\frac{du_4}{dt} - u_3\,(u_4 - \tfrac{1}{2}) = \frac{1}{R}\,(u_3 - 2u_4)\,. \end{aligned} \tag{13}$$

A closed-form solution of equation (13) is possible, if we assume the shock to be symmetric about $u_3 = \frac{1}{2}$. Thus, we get

$u_2 + u_4 = 1$ and

$$\frac{du_2}{dt} + \left(\frac{2}{R} - \frac{1}{2}\right) u_2 = \frac{3}{2R} - \frac{1}{4}\,, \tag{14a}$$

$$\frac{du_4}{dt} + \left(\frac{2}{R} - \frac{1}{2}\right) u_4 = \frac{1}{2R} - \frac{1}{4}\,. \tag{14b}$$

Integration of Equation (14a) gives

$$u_2 = u_2(0)\, e^{-\left(\frac{2}{R} - \frac{1}{2}\right)t} + \frac{\frac{3}{2R} - \frac{1}{4}}{\frac{2}{R} - \frac{1}{2}} \left\{1 - e^{-\left(\frac{2}{R} - \frac{1}{2}\right)t}\right\} \tag{15}$$

Once again the solution grows with time, unless $R > 4$. From the two cases considered, it appears that if the mid-point symmetry is not fixed the solution will diverge for $R > 2$ while a converged symmetric solution may be possible for $2 < R < 4$ if the mid-point value $u_3 = \frac{1}{2}$ is fixed. The non-conservative calculations of Reference [3] confirm the validity of this inequality for the cases considered.

2.2.2 *Conservation form*

A similar procedure can be carried out for the conservative difference form of Burgers equation.

(i) $N = 3$: In this case, the governing system reduces to

$$\frac{du_2}{d\tau} + \frac{2}{R} u_2 = \frac{1}{R} .$$

Integration gives

$$u_2(\tau) = \tfrac{1}{2} + \{u_2(0) - \tfrac{1}{2}\} \exp(-2\tau/R) . \tag{16a}$$

Significantly, this system leads to a steady converged state with $u_2(\tau) = \frac{1}{2}$ for all R.
(ii) $N = 5$: The governing equations are

$$\frac{du_2}{d\tau} + \frac{u_3(u_3-1)}{2} = \frac{1}{R}(u_3 - 2u_2 + 1) ,$$

$$\frac{du_3}{d\tau} + (u_4 - u_2)\frac{u_4 + u_2 - 1}{2} = \frac{1}{R}(u_4 - 2u_3 + u_2) ,$$

$$\frac{du_4}{d\tau} - \frac{u_3(u_3-1)}{2} = \frac{1}{R}(u_3 - 2u_4) .$$

Two steady-state solutions are possible,

$$u_2 + u_4 = 1 \text{ and } u_4 - u_2 = 2/R .$$

The symmetric solution characterized by $u_2 + u_4 = 1$ is stable for all R. The non-symmetric solution is given by:

$$u_3 = \left\{1 \pm \sqrt{1 - 4\left(4/R^2 - \frac{1}{R}\right)}\right\} \Big/ (2) .$$

$$u_2 = (1 + 2u_3 + 4/R)/(4) \tag{16b}$$

$$u_4 = (1 + 2u_3 - 4/R)/(4) .$$

This solution exists only for $R > 2.5$. In order to investigate the stability properties, we perturb about (16b) and look for the solutions of the type, $\exp(\lambda\tau)$. This leads to the following dispersion relation

$$\lambda = \frac{-2}{R_e} \pm \sqrt{\frac{4}{R_e^2} - (2u_3^0 - 1)^2/4} .$$

Since λ always has a negative real part, the solution (16b), in the steady state, is stable for $R \geqslant 2.5$. It should be pointed out that the solution with enforced symmetry (i.e., $u_3^0 = \frac{1}{2}$) is stable for all cell Reynolds numbers. This has been numerically tested for $\nu = 10^{-5}$ or $R = 10^5$. The resulting solution has oscillations but is stable. These oscillations can be eliminated or reduced in amplitude by applying the filters of the type discussed previously.

2.2.3 *Optimal filtering*

We recall that the different filters are defined by

$$\tilde{u}_j = \frac{u_{j+1} + u_{j-1} + ku_j}{2+k} \; ; \tag{17}$$

for $k = 0$, we recover the trapezoidal filter or conservation form; for $k = \infty$, non-conservation form is recovered; $k = 2$ corresponds to the Shuman filter and $k = -1$ is a noteworthy case. The finite-difference form of Burgers equation (2) with the filter (17) is a weighted average of non-conservation and conservation equations and is given as

$$\frac{du_j}{d\tau} + \left[\frac{u_{j+1} + u_{j-1} + ku_j}{2+k} - \tfrac{1}{2}\right](u_{j+1} - u_{j-1}) = \frac{1}{R}\,(u_{j+1} - 2u_j + u_{j-1})\,. \tag{18}$$

We shall examine the case where $N = 3$ in order to obtain an optimum k value for (18). For $N = 3$, (18) becomes

$$\frac{du_2}{d\tau} - \left(\frac{1 + ku_2}{2+k} - \tfrac{1}{2}\right) = \frac{1}{R}\,(1 - 2u_2)\,,$$

so that

$$u_2(\tau) = \tfrac{1}{2} + [u_2(0) - \tfrac{1}{2}]\exp\left(\frac{k}{2+k} - \frac{2}{R}\right)\tau\,. \tag{19}$$

For large R, $k > 0$ *has a destabilizing influence* and $k \leqslant 0$ *has a stabilizing effect*. We recall that the filtering introduces an artificial viscosity $\sim \Delta x^2 u_x/([2\,(2+k)])$, when $-2 < k < 0$. For $k < -2$, the sign of this viscosity changes and consequently the filter is no longer useful. From the previous analysis of non-conservation or conservation solutions, it is known that oscillatory behavior occurs when $R > 2$. These oscillations can be eliminated by reducing the grid size and therefore the local value of R. The filter (17) can also accomplish this without grid reduction. For large $R(>> 2)$ a minimum amount of filtering is required in order to obtain smooth solutions. The degree of filtering, as characterized by the value of k, should be such that the changes in u_j are confined to only one grid point. This amounts to incorporating artificial viscosity such that the cell Reynolds number based on the 'effective viscosity' does not exceed 2; the optimum choice for k can be seen by comparing (19) with (16a) such that $R_{eff} = 2$ or

$$\frac{1}{R} = \frac{1+k}{2+k}$$

This relation has also been obtained by Cheng and Shubin [9] from different considerations. Results for several filters are presented in the following section.

3. Results

Numerical solutions using various filters were obtained for a variety of Courant and cell Reynolds numbers. These do confirm the stability analysis of the previous section. For example, for large cell Reynolds number and Courant number, a converged solution for conservation form is obtained by enforcing the symmetry condition. This solution exhibits oscillations, characteristic of large cell Reynolds number flows. Without enforced symmetry, the shock may convect to the boundary. This corresponds to the second solution discussed previously. In the cases considered here, symmetric solutions are usually found for Courant numbers less than one. In these cases a completely symmetric solution is not achieved. The shock continues to move with an extremely small velocity. However, within a prescribed tolerance, the solution can be considered to be converged. Calculations for a variety of cell Reynolds numbers ranging from 2.4 to 50,000 and various filters characterized by $-1 \leqslant k \leqslant 1$, were carried out by Taverna and Busch [10]. For a given cell Reynolds number, an optimum filter was defined by a minimum error condition. Velocity (u) profiles with and without filters are shown on Figure 1.

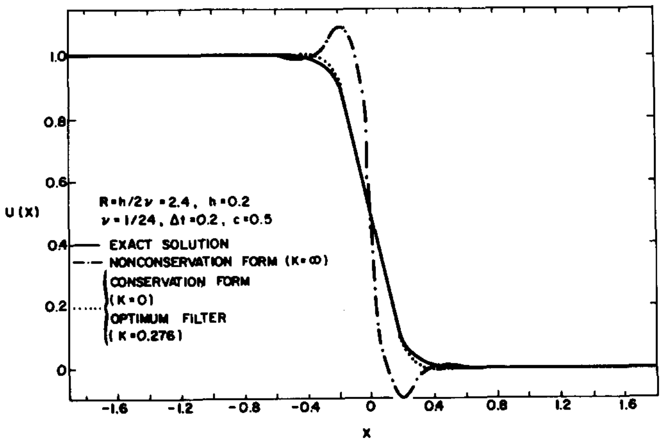

Figure 1a. Filtered Burgers equation: R = 2.4

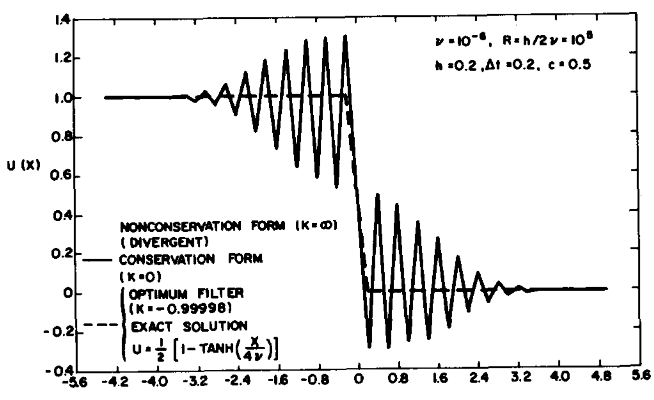

Figure 1b. Filtered Burgers equation: R = 100,000

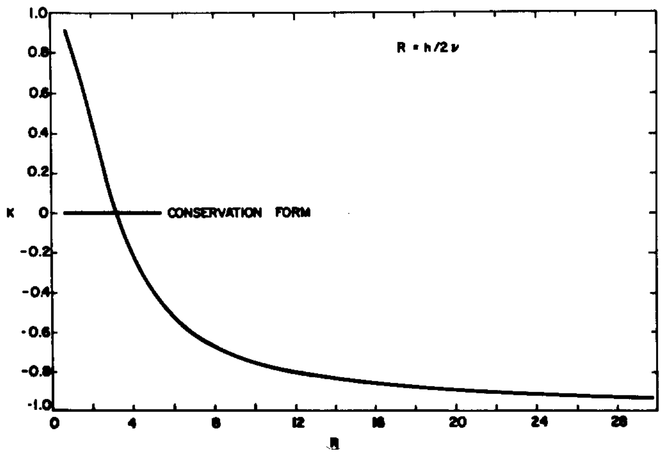

Figure 2. Optimum filter (k) vs. R

The optimum values as a function of R (Figure 2) are also reproduced from Reference [10]. It should be noted that with an appropriate filter, oscillations characteristic of large cell Reynolds numbers can be eliminated; formally, second order accuracy is still maintained. For $R \to \infty$, we note that k optimum $\to -1$.

4. Summary

1. Diagonal dominance problems associated with ICD methods can be eliminated by the KR scheme. Calculations with a linear Burgers equation confirm the analysis of Reference [2].

2. Stability problems arising in calculations with locally large cell Reynolds numbers are found only for non-linear equations and are due to the form of spatial differencing of the convective terms. This instability and associated oscillations can be eliminated by appropriate filtering.

3. A non-uniqueness of the conservation form of the difference equations is described. The second (non-physical) solution is encountered numerically only for large Courant numbers and $R > 2$.

4. Finally the results of [1-3] have been confirmed by numerical experimentation as well as some approximate stability analysis.

5. The present analysis has been confined to the stationary solution of Burgers equation, although some results for boundary layers are given in the Appendix. For more general equations, the nature of the optimum filter may vary from that obtained here.

Appendix. Blasius equation

The flat-plate boundary layer in similarity variables is governed by

$$u_{\eta\eta} + fu_\eta = 0, \; f_\eta = u, \qquad \text{(non-conservation form)}$$

or

$$u_{\eta\eta} + (fu)_\eta - u^2 = 0. \qquad \text{(conservation form)}$$

The finite-difference form using weighted averaging with $h = \Delta\eta$ is given by

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \frac{f_{j+1} + kf_j}{2(1+k)h} u_{j+1} - \frac{kf_j + f_{j-1}}{2(1+k)h} u_{j-1} \frac{-u_j^2}{1+k} = 0. \tag{A.1}$$

Clearly $k = 0$ and $k = \infty$ lead to the conservation and non-conservation forms. In order to examine the effect of filtering, the truncation error of equation (A.1) is investigated. Taylor series expansion about the jth grid point leads to

$$\left(1 + \frac{2h^2u}{3(1+k)}\right) u_{\eta\eta} + \left(f + \frac{h^2u_\eta}{2(1+k)}\right) u_\eta = \frac{-h^2}{6} [fu_{\eta\eta\eta} + \tfrac{1}{2}u_{\eta\eta\eta\eta}] \tag{A.2}$$

The second-order accuracy of the numerical scheme is retained. The additional truncation error arising out of weighted averaging is shown as coefficients of the convective as well as diffusive terms. It may be seen that the convective and diffusive modification can be made small near the surface by taking a fine grid. However, near the edge of the boundary layer, where large h values can lead to a deterioration of the solution resulting in oscillations, the numerical viscosity of the filter can be quite large and thus the oscillations are suppressed. The effect of filtering is therefore to incorporate damping where it is needed the most. The filtering effect is largest for $-1 < k < 0$ for these equations. In the present context of filtering, conservation form provides a significant amount of artificial viscosity, so that solutions with large mesh sizes are possible. For example, a converged solution with $h = 6$ (the boundary layer thickness is about 3.5) can be obtained; non-conservation solutions are no longer possible when $h > 2$. This artificial viscosity provides a thickening of the boundary-layer and consequently a reduction of the wall shear. Smaller grids near the wall surface are required to eliminate this accuracy problem. Optimally, it would appear that non-conservation form should be used near the surface and conservation form in the outer portion of the boundary layer. This corresponds to a variable filter.

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